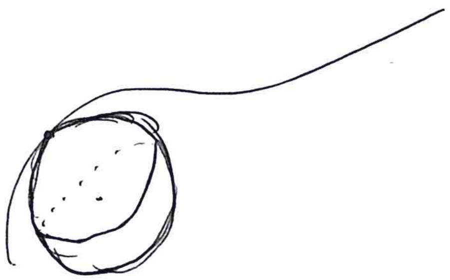
Tutorial 2 DG

1. Osculating sphere (Thm 2.10 in the Textbook)

In $\mathbb{R}^3$,



Thm:

c = Frenet curve in $\mathbb{R}^3$ with $\tau(s_0) \neq 0$. (parametrized by arc-length)

The sphere with center $c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) - \frac{\kappa'(s_0)}{\tau(s_0)\kappa^2(s_0)} e_3(s_0)$

passing through $c(s_0)$

has a point of contact with the curve $c(s)$ at the point s_0 of 3^{rd} order. This sphere is uniquely determined by these properties and is called the osculating sphere.

Pf:

Assume centre $m(s_0) = c(s_0) + \alpha e_1(s_0) + \beta e_2(s_0) + \gamma e_3(s_0)$,

α, β, γ are to be determined.

Consider

$$r(s) = \langle m - c(s), m - c(s) \rangle \quad , \quad m = m(s_0)$$

$$r' = -2 \langle m - c(s), c'(s) \rangle$$

$$r'' = -2 \langle m - c(s), c''(s) \rangle + 2 \langle c'(s), c'(s) \rangle$$

$$r''' = -2 \langle m - c(s), c'''(s) \rangle + 2 \langle c'(s), c''(s) \rangle + 4 \langle c'(s), c''(s) \rangle$$

$$= -2 \langle m - c(s), c'''(s) \rangle + 6 \langle c'(s), c''(s) \rangle$$

note that $\langle c', c'' \rangle = 0$, $r''' = -2 \langle m - c(s), c'''(s) \rangle$.

The optimal contact of the sphere with $c(s)$ means that as many derivatives of $r(s)$ as possible vanish at $s = s_0$. i.e

$$r'(s_0) = 0 \iff \langle m - c(s_0), c'(s_0) \rangle = 0$$

$$\iff \langle m - c(s_0), e_1(s_0) \rangle = 0$$

$$\iff \alpha = 0.$$

$$r''(s_0)=0 \iff \langle m-c(s_0),\, c''(s_0)\rangle - \langle c'(s_0), c'(s_0)\rangle = 0$$

$$e_2 = \frac{c''}{\kappa}$$

$$\iff \langle \beta e_2 + r e_3,\, \kappa e_2\rangle = 1$$

$$\iff \beta = \frac{1}{\kappa(s_0)}$$

$$r'''(s_0)=0 \iff \langle m-c(s_0),\, c'''(s_0)\rangle = 0$$

$$\iff \left\langle \frac{1}{\kappa} e_2 + r e_3,\, \kappa' e_2 + \kappa(-\kappa e_1 + \tau e_3)\right\rangle = 0$$

$$\iff \frac{\kappa'}{\kappa} + r\kappa\tau = 0$$

$$\iff r = -\frac{\kappa'(s_0)}{\kappa^2(s_0)\tau(s_0)}.$$

So $m(s_0) = c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) - \frac{\kappa'(s_0)}{\kappa^2(s_0)\tau(s_0)} e_3(s_0)$.

Centre $= m(s_0)$, Radius $= \sqrt{\frac{1}{\kappa^2(s_0)} + \frac{\kappa'^2(s_0)}{\kappa^4(s_0)\tau^2(s_0)}}$ $\Rightarrow$ Sphere S_{s_0}

pass through $c(s_0)$, $r'(s_0)=0 \Rightarrow c'(s_0)$ is inside the tangent plane of S_{s_0} at s_0, $\exists$ curve $\delta(s)$ on S_{s_0} s.t. $\delta(s_0)=c(s_0)$, $\delta'(s_0)=c'(s_0)$

$r''(s_0) = 0 \quad \Rightarrow \quad \gamma''(s_0) = c''(s_0)$

$r'''(s_0) = 0 \quad \Rightarrow \quad \gamma'''(s_0) = c'''(s_0).$

So we say this sphere has a pt of contact with the curve $c(s)$ at s_0 of 3^{rd} order.

2. Slope lines (Thm 2.11 in the Textbook)

For a Frenet curve in $\mathbb{R}^3$, the following conditions are equivalent:

(i) $\exists\ v \in \mathbb{R}^3 \setminus \{0\}$ with the property that $\langle e_1, v \rangle$ is constant.

(ii) $\frac{\tau}{\kappa}$ is constant.

We call such curve slope line.

Pf: (i) $\Rightarrow$ (ii)

We may write $\langle e_1, v \rangle = |v| \cos\theta$ for some constant θ

Differentiation $\Rightarrow$ $0 = \langle e_1', v \rangle$

$$= \kappa \langle e_2, v \rangle$$

$\Rightarrow$ $\langle e_2, v \rangle = 0$ since $\kappa \neq 0$

$\Rightarrow$ $V = \langle v, e_1 \rangle e_1 + \langle v, e_3 \rangle e_3$

$= |v| \cos\theta \, e_1 + |v| \sin\theta \, e_3$ (Change θ to $-\theta$ if necessar

Differentiation $\Rightarrow$

$$0 = |v| \cos\theta \, e_1' + |v| \sin\theta \, e_3'$$

$\Rightarrow$

$$0 = \cos\theta \, \kappa e_2 + \sin\theta \, (-\tau e_2)$$

$$= (\kappa \cos\theta - \tau \sin\theta) e_2$$

$\Rightarrow$ $\kappa \cos\theta - \tau \sin\theta = 0$

$\Rightarrow$ $\dfrac{\tau}{\kappa} = \cot\theta$ ($\theta \neq 0$ otherwise $\kappa = 0$ i.e. lin

(ii) $\Rightarrow$ (i) If $\dfrac{\tau}{\kappa} =$ constant, $\exists \, \theta$ s.t

$$\frac{\tau}{\kappa} = \cot\theta .$$

Def $V = \cos\theta\, e_1 + \sin\theta\, e_3$

$$V' = \cos\theta\, e_1' + \sin\theta\, e_3'$$

$$= (k\cos\theta - \tau\sin\theta)\, e_2$$

$$= 0$$

$$\langle e_1, V\rangle = \cos\theta = \text{constant}\ !$$

$\square$

Rmk: Helix is an example.

Please also read Thm 2-11 in the textbook since I provide a different proof.

3. Suggested exercise 3 in HW1.

The Frenet two-frame of a plane curve with given curvature function $k(s)$ can be described by the exponential series for the matrix

$$\begin{pmatrix} 0 & \int_0^s k(t)dt \\ -\int_0^s k(t)dt & 0 \end{pmatrix}$$

i.e $\begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & \int_0^s k(t)dt \\ -\int_0^s k(t)dt & 0 \end{pmatrix}^i .$

Pf: Given a curvature function $k(s)$, assume

$$e_1 = (\cos(\alpha(s)), \sin(\alpha(s)))$$

then $e_2 = (-\sin(\alpha(s)), \cos(\alpha(s)))$

$$ke_2 = e_1' = \alpha'(-\sin(\alpha(s)), \cos(\alpha(s))) = \alpha' e_2$$

$\Rightarrow \quad k = \alpha'$

$\Rightarrow \quad \alpha(s) = \int_0^s k(t)\, dt$ if $\alpha(0) = 0$ i.e $e_1(0) = (1, 0)$

$\Rightarrow \quad e_1 = \left(\cos\left(\int_0^s k\right), \sin\left(\int_0^s k\right)\right)$

$e_2 = \left(-\sin\left(\int_0^s k\right), \cos\left(\int_0^s k\right)\right)$

i.e

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}^n$$

(note that $\begin{pmatrix} 0 & +x \\ -x & 0 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$)

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} - \cdots$$